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ON THE SOLUTIONS OF CONVECTION-DIFFUSION PROBLEMS USING ELZAKI HOMOTOPY PERTURBATION METHOD

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Abstract: In this paper, the Elzaki Homotopy Perturbation Method which is the combination of Elzaki Transform and Homotopy Perturbation Method was applied in solving convection-diffusion problems which emerges in physical phenomena where particles and other quantities were transferred within a physical system as a result of diffusion and convection. Three examples were solved by this proposed method to obtain the approximate solutions of respective examples and the results obtained were compared with other existing exact solutions. It was confirmed that the result of the proposed method agrees perfectly with results obtained by other methods.

Keywords: Elzaki transform, Homotopy Perturbation method, diffusion, partial differentiation equation

Introduction

Convection-diffusion equations are the mixture of the diffusion and convection equations which specify the physical occurrence by which particles, mass, energy, and several other physical quantities are moved into the physical system through the process of diffusion and convection.

Generally, the convection-diffusion equation is given as:

$$\frac{\partial u}{\partial t} = \nabla.(D.\nabla u) - \nabla.(\vec{v}u) + R \qquad 1$$

Let u represents the variable of interest, D represents the diffusivity, $\vec{v}$ represents average velocity, R serves as the source and sink of the quantity of u , ∇ is the gradient, and $\nabla.$ is the divergence.

Because of the significance of the convection-diffusion equations, over the years several methods have been used to solve convection-diffusion problems. These include Variational Iteration Method (Liu and Zhao, 2010), Adomian Decomposition Method (Momani, 2007), Decomposition Method (Yee, 1993), He's Homotopy Perturbation Method (Momani and Yildirim, 2010), Bessel Collocation Method (Yuzbasi and Sahin, 2013), Finite difference method (Faghri *et al.*, 2007), Mahgoub Transform (Dehinsilu *et al.*, 2020), B-Spline Collocation Method (Kadalbajoo *et al.*, 2008), Wavelet-Galerkin Method (El-Gamel, 2012). The majority of these aforementioned methods have their drawbacks such as the calculations of Adomian polynomials.

Recently, several numerical techniques have been applied to solve diverse physical problems such as continuous population models for single and interacting species (Yuzbasi, 2011), nonlinear Lane Emden type equations (Yuzbasi, 2011), Hantavirus infection model (Yuzbasi and Sezer, 2013). The Homotopy perturbation techniques were first developed by He (He, 1999; 2003; 2004; 2006)for solving several classes of initial and boundary value problems. Several kinds of research have been carried out on HPM by many researchers to handle the nonlinearity which arises in engineering sciences (Xu, 2007; Yildirim, 2009; Ganji, 2006; Odibat, 2008). Such as Laplace Transform Method and Homotopy Perturbation Method (Gupta, Kumar and Singh, 2015), Homotopy Method, and Elzaki Transform (Elzaki and Biazar, 2013), Elzaki Homotopy Perturbation Method (2014), Homotopy Perturbation Method and Aboodh Transform (Aboodh, 2017).

The major aim of this work is to apply a modification of Homotopy Perturbation Method to defeat the deficiency. Here, we apply Elzaki Homotopy Perturbation Method to solve the convection-diffusion problems. The proposed EHPM yield solutions which agree with the exact solutions of the problems considered. Three problems were considered to confirm the applicability and effectiveness of the Elzaki Homotopy Perturbation Method.

Materials and Methods

EHPM is a combination of Elzaki Transform and Homotopy Perturbation Method. EHPM is applied to the general nonlinear partial differentiation of the form:

$$Du(x,t) + Ru(x,t) + Nu(x,t) = g(x,t) \qquad 2$$

$$u(x,t) = h(x), u_t(x,u) = f(x) \qquad 3$$

Where D is the linear differentiation operator of second order $D = \frac{\partial^2}{\partial t^2}$; R is the linear differential operator; N is the general nonlinear differential operator and $g(x,t)$ represents the source term

Taking the Elzaki Transform of equation (2) yields,

$$E\left[Du(x,t) + Ru(x,t) + Nu(x,t) = E[g(x,t)]\right]$$

$$E[Du(x,t)] + E[Ru(x,t)] + E[Nu(x,t)] = E[g(x,t)] \qquad 4$$

Applying the differential property of the Elzaki Transform yields

$$E[u(x,t)] = v^2h(x) + v^3f(x) - v^2E[Ru(x,t)] \qquad 5$$

Taking the Inverse Elzaki Transform on both sides of equation (5)

$$u(x,t) = E^{-1}\left[v^2h(x) + v^3f(x) + v^2E[g(x,t)] - E^{-1}\left[v^2E[Ru(x,t) + Nu(x,t)]\right]\right] \qquad 6$$

$$= G(x,t) - E^{-1}\left[v^2E[Ru(x,t) + Nu(x,t)]\right] \qquad 7$$

$G(x,t)$ is the resulting term from the source term with the associated initial conditions.

We then apply Homotopy Permutation Method

$$u(x,t) = \sum_{n=0}^{\infty} p^n u_n(x,t) \qquad 8$$

The nonlinear term can be decomposed as

$$Nu(x,t) = \sum_{n=0}^{\infty} p^n H_n(x,t) \qquad 9$$

He's polynomials. $H_n(u)$ are given by

$$H_n(u_0, u_1, \dots u_n) = \frac{1}{n!}\frac{\partial^n}{\partial p^n}\left[N\left(\sum_{i=0}^{\infty} p^i u_i\right)\right], p = 0, n = 0,1,2,3,\dots \qquad 10$$

Putting equations(8) and (9) into equation (7), we have

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = G(x,t) - p\left(E^{-1}\left[v^2E\left[R\sum_{n=0}^{\infty} p^n u_n(x,t) + \sum_{n=0}^{\infty} p^n H_n(u)\right]\right]\right) \qquad 11$$

Equation (11) is the combination of the Elzaki Transform and the Homotopy Perturbation Method by using He's polynomials.

Inspecting equation (11) by comparing the coefficient of the corresponding powers of p

$$p^0: u_0(x,t) = G(x,t)$$

$$p^1: u_1(x,t) = -E^{-1}\left[v^2E[Ru_0(x,t) + H_0(u)]\right]$$

$p^2: u_2(x,t) == -E^{-1}\left[v^2 E[Ru_1(x,t) + H_1(u)]\right]$
$p^3: u_3(x,t) == -E^{-1}\left[v^2 E[Ru_2(x,t) + H_2(u)]\right]$ 12

Results and Discussion
Here, we demonstrate the applicability of the Elzaki Homotopy Perturbation Method by applying it to solve some convection-diffusion problems

Example 1
Solve the Diffusion-Convection problem (Dehinsilu *et al.*, 2020)

$$\frac{\partial u}{\partial t} = \frac{\partial^2}{\partial x^2} - u \quad 13$$

With the initial condition $U(x,0) = x + e^{-x}$
Taking the Elzaki Transform on both sides of equation (13)

$$E\left[\frac{\partial u}{\partial t}(x,t)\right] = E\left[\frac{\partial^2}{\partial x^2} - u\right] \quad 14$$

$$T\frac{(x,u)}{v} - vu(x,0) = E\left[\frac{\partial^2}{\partial x^2} - u\right]$$

$$T[u(x,u)] = v^2 u(x,0) + vE\left[\frac{\partial^2}{\partial x^2} - u\right]$$

Substituting the initial condition

$$E[u(x,t)] = v^2(x + e^{-x}) + vE\left[\frac{\partial^2}{\partial x^2} - u\right]$$

Taking the Inverse Elzaki Transform,

$$u(x,t) = E^{-1}[v^2(x + e^{-x})] + E^{-1}\left[vE\left(\frac{\partial^2}{\partial x^2} - u\right)\right] \quad 15$$

Applying Homotopy Perturbation Method,

$$u(x,t) = \sum_{n=0}^{\infty} p^n u_n(x,t) \quad 16$$

Substituting equation (16) into (15)

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = (x + e^{-x}) + pE^{-1}\left[vE\left(\sum_{n=0}^{\infty} p^n u_n\, xx(x,t)\right) - L\left(\sum_{n=0}^{\infty} p^n u_n\,(x,t)\right)\right] \quad 17$$

Inspecting equation (17) and comparing the coefficients of the corresponding powers of p

$p^0: u_0(x,t) = x + e^{-u}$
$p^1: u_1(x,t) = -xt$
$p^2: u_1(x,t) = \frac{xt^2}{2!}$ 18
$\vdots$

Other components of $u(x,t)$ can be obtained. The solution of equation (22) can be written by following the same procedure The series solution is obtained as:

$$u(x,t) = u_0 + u_1 + u_2 + u_3 + \cdots \quad 19$$

$$u(x,t) = x + e^{-u} + (-xt) + \frac{xt^2}{2!} + \left(-\frac{xt^3}{3!}\right) + \cdots$$

$$= e^{-x} + x\left(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \cdots\right) \quad 20$$

Equation (20) which is the solution of equation (13) can be written in a closed-form as

$$u(x,t) = e^{-x} + xe^{-t} \quad 21$$

Equation (21) agrees with the exact solution and it is the same as the result obtained in (Dehinsilu *et al.*, 2020)

Example 2
Consider the following diffusion-convection problem (Gupta *et al.*, 2015)

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - \frac{1}{4}u, x, t \in \mathbb{R} \quad 22$$

With the initial condition $u(x,0) = \frac{1}{2}x + e^{-\frac{x}{2}}$
Taking the Elzaki Transform on both sides of equation (22)

$$\frac{\partial u}{\partial t} = E\left[\frac{\partial^2 u}{\partial x^2} - \frac{1}{4}u\right] \quad 23$$

$$\frac{T(x,u)}{v} = vu(x,0) = E\left[\frac{\partial^2 u}{\partial x^2} - \frac{1}{4}u\right]$$

Substituting the initial condition

$$E[u(x,t) = v^2\left(\frac{1}{2}x + e^{-\frac{x}{2}}\right) + vE\left[\frac{\partial^2 u}{\partial x^2} - \frac{u}{4}\right] \quad 24$$

Taking the Inverse Elzaki Transform on both sides of equation (24)yield

$$u(x,t) = \left(\frac{x}{2} + e^{-\frac{x}{2}}\right) + E^{-1}\left[vE\left[\frac{\partial^2 u}{\partial x^2} - \frac{u}{4}\right]\right] \quad 25$$

Applying Homotopy Perturbation Method

$$u(x,t) = \sum_{n=0}^{\infty} p^n u_n\,(x,t) \quad 26$$

Substituting equation (26) into (25)

$$\sum_{n=0}^{\infty} p^n u_n\,(x,t) = \left(\frac{x}{2} + e^{-\frac{x}{2}}\right) + pE^{-1}\left[vE\left[\sum_{n=0}^{\infty} p^n u_{nxx}\,(x,t) - \frac{1}{4}\sum_{n=0}^{\infty} p^n u_n\,(x,t)\right]\right] \quad 27$$

Comparing the coefficients of the corresponding powers of P, we have

$p^0: u_0(x,t) = \frac{x}{2} + e^{-\frac{x}{2}}$
$p^1: u_1(x,t) = \frac{x}{2}\left(-\frac{t}{4}\right)$
$p^2: u_2(x,t) = \frac{x}{2}\frac{\left(\frac{t}{4}\right)^2}{2!}$
$p^3: u_3(x,t) = \frac{x}{2}\frac{\left(-\frac{t}{4}\right)^3}{3!}$ 28
$\vdots$

Other components of $u(x,t)$ can be obtained. The solution of equation (22) can be written by following the same procedure in the form

$$u(x,t) = u + u_1 + u_2 + u_3 + \cdots \quad 29$$

$$= \left(\frac{x}{2} + e^{-\frac{x}{2}}\right) + \frac{x}{2}\left(-\frac{t}{4}\right) + \frac{x}{2}\frac{\left(\frac{t}{4}\right)^2}{2!} + \frac{x}{2}\frac{\left(-\frac{t}{4}\right)^3}{3!} + \cdots$$

$$u(x,t) = \frac{x}{2} + e^{-\frac{x}{2}} - \frac{x}{2}\left(-\frac{t}{4}\right) + \frac{x}{2}\frac{\left(\frac{t}{4}\right)^2}{2!} - \frac{x}{2}\frac{\left(-\frac{t}{4}\right)^3}{3!} + \cdots$$

$$u(x,t) = e^{-\frac{x}{2}} + \frac{x}{2}\left(1 - \frac{t}{4} + \frac{\left(\frac{t}{4}\right)^2}{2!} - \frac{\left(-\frac{t}{4}\right)^3}{3!} + \cdots\right) \quad 30$$

Equation (30) which is the solution of equation (22) can be written in a closed-form as

$$u(x,t) = e^{-\frac{x}{2}} + \frac{x}{2}e^{-\frac{t}{4}} \quad 31$$

Equation (31) agrees with the exact solution of equation (22) and it is the same as the solution obtained (Gupta, Kumar, and Singh, 2015).

Example 3
Considering the following diffusion-convection problem (Gupta, Kumar and Singh, 2015)

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial x} + uu_x - u^2 + u, x, t \in \mathbb{R} \quad 32$$

With the initial conditions $u(x,0) = e^x$
Taking the Elzaki Transform on both sides of equation (32)

$$E\left[\frac{\partial u}{\partial t}\right] = E\left[\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial x} + uu_x - u^2 + u\right] \quad 33$$

$$T\frac{(x,t)}{v} - vu(x,0) = E\left[\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial x} + uu_x - u^2 + u\right]$$

$$T(x,v) = v^2 u(x,o) + vE\left[\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial x} + uu_x - u^2 + u\right]$$

$$E[u(x,t)] = v^2 u(x,o) + vE\left[\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial x} + uu_x - u^2 + u\right]$$

Substitution the initial condition, we have

$$E[u(x,t)] = v^2 e^x + vE\left[\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial x} + uu_x - u^2 + u\right]$$

Taking Inverse Elzaki Transform

$$u(x,t) = e^x + E^{-1}\left[vE\left[\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial x} + uu_x - u^2 + u\right]\right] \quad 34$$

Applying Homotopy Perturbation Method

$$u(x,t) = \sum_{n=0}^{\infty} p^n u_n(x,t) \quad 35$$

Substituting equation (35) into equation (34)

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = e^x + pE^{-1}\left[vE\left[\sum_{n=0}^{\infty} p^n u_{nxx}\,(x,t) \sum_{n=0}^{\infty} p^n u_{nx}\,(x,t) + \sum_{n=0}^{\infty} p^n u_n(x,t) \sum_{n=0}^{\infty} p^n u_{nx}\,(x,t) - \left(\sum_{n=0}^{\infty} p^n u_n\right)^2 + \sum_{n=0}^{\infty} p^n u_n\right]\right] \quad 36$$

$$\sum_{n=0}^{\infty} p^n u_n\,(x,t) = e^x + pE^{-1}\left[vE\left(\sum_{n=0}^{\infty} p^n u_{nxx}\,(x,t)\right) - vE\left(\sum_{n=0}^{\infty} p^n u_{nx}\,(x,t)\right) + \right.$$

$vE\left(\sum_{n=0}^{\infty} p^n u_n(x,t)\right)\left(\sum_{n=0}^{\infty} p^n u_{nx}(x,t)\right) -$
$vE(\sum_{n=0}^{\infty} p^n u_n)^2 + vE(\sum_{n=0}^{\infty} p^n u_n)$ 37

Equating the coefficients of the corresponding power of p, we have:

$p^0: u_0(x,t) = e^x$
$p^1: u_1(x,t) = e^{xt}$
$p^2: u_2(x,t) = e^x \frac{t^2}{2!}$
$p^3: u_3(x,t) = e^x \frac{t^3}{3!}$ 38
$\vdots$

Other components can be obtained by following the same procedure.

The solution of equation (32) can be written in the form

$u(x,t) = u_0 + u_1 + u_2 + u_3 + \cdots$ 39

$u(x,t) = e^x + e^{xt} + e^x \frac{t^2}{2!} + \cdots$

$= e^x(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots)$ 40

Equation (40) which is the solution to equation (32) can be written in closed form as

$u(x,t) = e^{x+t}$ 41

Equation (41) agrees with the exact solution of equation (32) and it is the same as the solution obtained in (Gupta, Kumar and Singh, 2015).

Conclusion

In this work, Elzaki Homotopy Perturbation Method has been successfully applied to obtain the solutions of Convection-Diffusion problems. The major advantage of this method is that it does not involve any linearization or discretization and that it gives a more realistic series solutions of rapidly convergent sequences with elegantly computed terms. EHPM yields solutions that agree with the exact solutions of the problems considered. This shows that the method can be applied to other Convection-Diffusion problems.

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